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Simplified Large-N Limit in Stochastic Quantization

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In the large-N limit, $O(N)$-invariant models become exactly soluble due to factorization properties first emphasized by Migdal and Witten. It is shown that in this limit, the Langevin equation of stochastic quantization offers a direct and simple determination of the mass gap. The method is applied to different bosonic and fermionic models.

1. INTRODUCTION

The method of stochastic quantization, introduced by Parisi and Wu (1981), provides an alternative to standard methods of quantization in Euclidean quantum field theories. It is based on the purely classical Langevin equation well known from the theory of Brownian motion.

Stochastic quantization is currently widely used and is of particular interest to study some properties of quantum field theories in the large-N limit. This became apparent after some rather spectacular developments in lattice gauge theories (Eguchi and Kawai, 1982), and the proof of quenching properties associated to certain sets of fields (Bhanot *et al.*, 1982), dubbed master fields. Exact solutions exist for these master fields (Greensite, 1983; Greensite and Halpern, 1983). Alfaro (1984; Alfaro and Sakita, 1983) obtains them using ansätz where invariant quantities built out of reduced fields are independent of the Langevin noise.

Our purpose here is to recover the large-N limit directly from the Langevin equation using only the factorization property. In Section 2 we

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briefly review the formalism of stochastic quantization. Then the method is detailed in Section 3, with the $O(N)$ ϕ^4 and Gross–Neveu models as illustrating examples. It is shown further in Section 4 that it can be extended to cases with more than one interacting fields, a particular case being the Hubbard–Stratanovitch auxiliary field procedure. Scalar fields with constraint and the multiple-field model of Zinn-Justin (1991) are considered.

2. STOCHASTIC QUANTIZATION

According to Parisi and Wu (1981) the Euclidean path integral measure

$$\exp(-S_E[\phi]) \tag{1}$$

is considered as the equilibrium distribution of a stochastic process in an extra variable, the fictitious time t.

The evolution in this new variable is governed by a Langevin equation:

$$\frac{\partial\phi(x, t)}{\partial t} = -\frac{\delta S_E[\phi]}{\delta\phi(x, t)} + \eta(x, t) \tag{2}$$

Here S_E is the classical action and η is a white Gaussian noise, i.e., a random field with first moment zero and second moment

$$\langle\eta(x, t)\eta(x', t')\rangle_\eta = 2\delta(t - t')\delta^D(x - x') \tag{3}$$

The principal assertion concerns the limit $t \to \infty$: the equilibrium distribution of equation (1) is reached, and all equal-time correlation functions tend to the corresponding quantum Green functions, i.e.,

$$\langle\phi(x_1, t) \cdots \phi(x_n, t)\rangle_\eta \underset{t\to\infty}{=} \langle\phi(x_1) \cdots \phi(x_n)\rangle \tag{4}$$

For $O(N)$-invariant models, an interesting simplification occurs in the large-N limit: the factorization property which was first emphasized by Migdal (1980) and Witten (1980). Vacuum expectation values of $O(N)$-invariant products of operators become free of fluctuations, i.e.,

$$\left\langle\prod_i A_i\right\rangle \underset{N\to\infty}{=} \prod_i \langle A_i\rangle \tag{5}$$

This property is unmysterious when one recalls the classical nature of the saddle point solution of the partition function, which is exact in the limit $N \to \infty$.

Alfaro (1984) has generalized this property to the stochastic quantization scheme. It reads

$$\left\langle \prod_i A_i[\phi] \right\rangle_\eta \underset{N\to\infty}{=} \prod_i \langle A_i[\phi]\rangle_\eta \tag{6}$$

In particular,

$$\langle A^2[\phi]\rangle_\eta \underset{N\to\infty}{=} (\langle A[\phi]\rangle_\eta)^2 \tag{7}$$

Since the noise measure is positive, we have then

$$\lim_{N\to\infty} A[\phi] = \langle A[\phi]\rangle_\eta$$

Hence in the large-N limit, every $O(N)$-invariant functional is noise independent.

3. LARGE-N LIMIT OF THE ϕ^4 AND GROSS–NEVEU MODEL

The Euclidean stochastic action of the $O(N)$ ϕ^4 model reads

$$S_{[\phi]} = \int d^Dx\, dt \left[\frac{1}{2}\,\partial_\mu\phi(x, t)\partial_\mu\phi(x, t) + \frac{1}{2}\, m^2\phi^2(x, t) + \frac{\lambda}{4!N}\,\phi^4(x, t)\right] \tag{8}$$

where

$$\phi^2(x, t) = \sum_{i=1}^{N} \phi_i(x, t)\phi_i(x, t)$$

The Langevin equation is then

$$\frac{\partial\phi_i(x, t)}{\partial t} = (\Box - m^2)\phi_i(x, t) - \frac{\lambda}{3!N}\,\phi^2(x, t)\phi_i(x, t) + \eta_i(x, t) \tag{9}$$

In this context the factorization property implies

$$\phi^2(x, t) \underset{N\to\infty}{=} \langle\phi^2(x, t)\rangle_\eta = \sigma(t) \tag{10}$$

where translational invariance of the Green functions has been used.

Hence in the only relevant limit of asymptotic times, $\sigma(t)$ tends to its equilibrium value σ_0, and in this regime of large N and infinite time, the Langevin equation is

$$\frac{\partial\phi_i(x, t)}{\partial t} = (\Box - m^2)\phi_i(x, t) - \frac{\lambda}{3!N}\,\sigma_0\phi_i(x, t) + \eta_i(x, t) \tag{11}$$

where

$$\lim_{t\to\infty} \sigma(t) = \sigma_0$$

This linear equation can be solved in the usual way. It is easy to see that the initial conditions disappear in the $t \to \infty$ limit and the two-point correlation function is obtained as

$$\lim_{\substack{t\to\infty \\ N\to\infty}} \left\langle \sum_{i=1}^{N} \phi_i(x, t)\phi_i(x, t) \right\rangle_\eta = \left\langle x \left| \frac{N}{-\Box + m^2 + (\lambda/6N)\sigma_0} \right| x \right\rangle = \sigma_0 \tag{12}$$

With

$$\Sigma_0 = m^2 + \frac{\lambda}{6N}\sigma_0 \tag{13}$$

equation (12) reads

$$\Sigma_0 = m^2 + \frac{\lambda}{6}\int \frac{d^Dp}{(2\pi)^D}\frac{1}{p^2 + \Sigma_0} \tag{14}$$

This is the well-known gap equation for this model.

This straightforward method can also be used for fermionic models. The most simple example is the Gross–Neveu model. In this case the Euclidean stochastic action is

$$S_{[\bar\psi]} = \int d^2x\, dt \left[-\bar\psi(x, t)\partial\!\!\!/\psi(x, t) + \frac{g^2}{2N}(\bar\psi(x, t)\psi(x, t))^2 \right] \tag{15}$$

where spins and internal symmetry indexes have been omitted. We have now two "conjugate" Langevin equations:

$$\left[\begin{array}{l} \dfrac{\partial\psi_i(x, t)}{\partial t} = \overrightarrow{\partial\!\!\!/}\, \psi_i(x, t) - \dfrac{g^2}{N}(\bar\psi(x, t)\psi(x, t))\psi_i(x, t) + \eta_i(x, t) \\ \dfrac{\partial\bar\psi_i(x, t)}{\partial t} = +\bar\psi(x, t)\, \overleftarrow{\partial\!\!\!/} - \dfrac{g^2}{N}\bar\psi_i(x, t)(\bar\psi(x, t)\psi(x, t)) + \bar\eta_i(x, t) \end{array}\right. \tag{16}$$

In the large-N limit, using as before factorization and translational invariance, we obtain

$$\lim_{N\to\infty} \sum_{i=1}^{N} \psi_i(x, t)\bar\psi_i(x, t) = \sigma(t) \tag{17}$$

Focusing on asymptotic time behavior, where

$$\lim_{t\to\infty} \sigma(t) = \sigma_0$$

we deduce

$$\lim_{\substack{N\to\infty \\ t\to\infty}} \left\langle \sum_{i=1}^{N} \psi_i(x, t)\bar{\psi}_i(x, t) \right\rangle_\eta = \sum_{\alpha=1}^{2} \left\langle x \left| \left(\frac{N}{-\not{\partial} + (g^2/N)\sigma_0} \right)_{\alpha\alpha} \right| x \right\rangle = \sigma_0 \tag{18}$$

Here the sum is on the spin indices.

With

$$\Sigma_0 = \frac{ig^2}{N} \sigma_0 \tag{19}$$

the usual gap equation for the Gross–Neveu model is retrieved:

$$1 + 2g^2 \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + \Sigma_0^2} = 0 \tag{20}$$

4. LARGE-N LIMIT FOR MODELS COUPLING BETWEEN DIFFERENT FIELDS

The method of the preceding section can be adapted to take into account the presence of multiple fields. It enables us to recover the large-N behavior without any supplementary difficulties.

4.1. ϕ^4 and Gross–Neveu Models with Auxiliary Fields

The well-known Hubbard–Stratanovitch stratagem, which introduces an auxiliary field $\sigma(x)$, renders the partition function Gaussian in the original fields. It is then possible to perform the integral over these fields. In stochastic quantization the auxiliary field acquires a fictitious time dependence and the effective action of the ϕ^4 model now reads

$$S_{\text{eff}}[\phi, \sigma] = \int d^Dx\, dt \left[\frac{1}{2} \partial_\mu\phi(x, t)\partial_\mu\phi(x, t) + \frac{1}{2} m^2\phi^2(x, t) + \frac{i\sqrt{\lambda}}{6!} \sigma(x, t)\phi^2(x, t) + \frac{N}{6!} \sigma^2(x, t) \right] \tag{21}$$

$\phi(x, t)$ and $\sigma(x, t)$ are then considered as two coupled stochastic fields governed by the following Langevin equations:

$$\begin{cases} \dfrac{\partial \phi_i(x, t)}{\partial t} = (\Box - m^2)\phi_i(x, t) - i \dfrac{\sqrt{\lambda}}{6} \sigma(x, t)\phi_i(x, t) + \eta_i(x, t) & (22) \\ \dfrac{\partial \sigma(x, t)}{\partial t} = -\dfrac{N}{12} \sigma(x, t) - i \dfrac{\sqrt{\lambda}}{12} \phi^2(x, t) + \theta(x, t) & (23) \end{cases}$$

where θ and η_i are $(N + 1)$ independent Gaussian white noises. If we take the mean value of (23) at the equilibrium limit $t \rightarrow \infty$, we find

$$0 = -\frac{N}{12} \langle \sigma(x) \rangle_{\eta\theta} - i \frac{\sqrt{\lambda}}{12} \langle \phi^2(x) \rangle_{\eta\theta} \tag{24}$$

Since the ϕ-quantum Green functions are translational invariant, we must have

$$\lim_{t \to \infty} \langle \phi^2(x, t) \rangle_{\eta\theta} = -i \frac{\sqrt{\lambda}}{N} \lim_{t \to \infty} \langle \sigma(x, t) \rangle_{\eta\theta} = \sigma_0 \tag{25}$$

Following Section 3, the factorization property for $O(N)$-invariant form implies

$$\begin{cases} \phi^2(x, t) \underset{N \to \infty}{=} \langle \phi^2(x, t) \rangle_{\eta\theta} \\ \sigma(x, t) \underset{N \to \infty}{=} \langle \sigma(x, t) \rangle_{\eta\theta} \end{cases} \tag{26}$$

Using these results in the large-N limit and for asymptotic time behavior, we can linearize and easily solve equation (24). We deduce then

$$\lim_{\substack{t \to \infty \\ N \to \infty}} \left\langle \sum_{i=1}^{N} \phi_i(x, t)\phi_i(x, t) \right\rangle_{\eta\theta} = \left\langle x \left| \frac{-N}{\Box - m^2 - i(\sqrt{\lambda}/6)\sigma_0} \right| x \right\rangle = \sigma_0 \tag{27}$$

or

$$\Sigma_0 = m^2 + \frac{\lambda}{6} \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 + \Sigma_0}$$

where

$$\Sigma_0 = m^2 + i \frac{\sqrt{\lambda}}{6} \sigma_0$$

For the Gross–Neveu model, the procedure is the same and we only give the main steps of the calculus.

The effective stochastic action reads

$$S_{\text{eff}}[\psi, \bar{\psi}, \sigma] = \int d^2x\, dt \left[-\bar{\psi}(x, t)\, \partial\!\!\!/ \psi(x, t) + \frac{N}{2g^2} \sigma^2(x, t) + i\bar{\psi}(x, t)\psi(x, t)\sigma(x, t) \right] \tag{28}$$

The leads to the Langevin equations

$$\begin{cases} \dfrac{\partial \psi_i(x, t)}{\partial t} = \overrightarrow{\partial\!\!\!/}\, \psi_i(x, t) - i\sigma(x, t)\psi_i(x, t) + \eta_i(x, t) \\ \dfrac{\partial \bar{\psi}_i(x, t)}{\partial t} = \bar{\psi}_i(x, t)\, \overleftarrow{\partial\!\!\!/} - i\bar{\psi}_i(x, t)\sigma(x, t) + \bar{\eta}_i(x, t) \\ \dfrac{\partial \sigma(x, t)}{\partial t} = -\dfrac{N}{g^2} \sigma(x, t) - i\bar{\psi}(x, t)\psi(x, t) + \theta(x, t) \end{cases} \tag{29}$$

Under the same arguments one finds

$$\lim_{\substack{N\to\infty \\ t\to\infty}} \left\langle \sum_{i=1}^{N} \psi_i(x, t)\bar{\psi}_i(x, t) \right\rangle_{\eta\theta} = \left\langle x \left| \frac{2i\sigma_0 N}{-(\partial\!\!\!/)^2 + \sigma_0^2} \right| x \right\rangle \tag{30}$$

and

$$0 = -\frac{N}{g^2} \sigma_0 - i \lim_{\substack{N\to\infty \\ t\to\infty}} \left\langle \sum_{i=1}^{N} \bar{\psi}_i(x, t)\psi_i(x, t) \right\rangle_{\eta\theta} \tag{31}$$

thence the gap equation

$$1 + 2g^2 \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + \Sigma_0} = 0 \tag{32}$$

where $\Sigma_0 = i\sigma_0$.

4.2. Constraint Models: Nonlinear σ and $\mathbb{C}P^{N-1}$ Models

In these models the constraint on a free $O(N)$ or $U(N)$ scalar field appears under the form of a Lagrange multiplier field in the action. For the nonlinear σ model it reads

$$S_{\text{eff}}[\phi, \lambda] = \int d^2x\, dt\, \frac{1}{2g^2} \left[\partial_\mu \phi(x, t) \partial_\mu \phi(x, t) + \lambda(x, t)(\phi^2(x, t) - 1) \right] \tag{33}$$

This Lagrange multiplier must be treated as a stochastic field with a specific Langevin equation

$$\begin{cases} \dfrac{\partial \phi_i(x, t)}{\partial t} = \dfrac{1}{g^2}(\Box \phi_i(x, t) - \lambda(x, t)\phi_i(x, t)) + \eta_i(x, t) & (34) \\ \dfrac{\partial \lambda(x, t)}{\partial t} = -\dfrac{1}{2g^2}(\phi^2(x, t) - 1) + \theta(x, t) & (35) \end{cases}$$

where θ and η_i are independent Gaussian white noises.

As before we can put

$$\lim_{\substack{N \to \infty \\ t \to \infty}} \lambda(x, t) = \lambda_0 \tag{36}$$

in (34), and we obtain

$$\lim_{\substack{N \to \infty \\ t \to \infty}} \left(\sum_{i=1}^{N} \langle \phi_i(x, t)\phi_i(x, t) \rangle_{\eta\theta} \right) = \left\langle x \left| \frac{-g}{\Box - \lambda_0} \right| x \right\rangle \tag{37}$$

In these limits (35) is nothing but the always true equation of constraints:

$$\sum_{i=1}^{N} \langle \phi_i(x, t)\phi_i(x, t) \rangle_{\eta\theta} = 1 \tag{38}$$

Then we arrive at the gap equation

$$\int \frac{d^2p}{(2\pi)^2} \frac{g}{p^2 + \lambda_0} = 1 \tag{39}$$

In the case of the $\mathbb{C}P^{N-1}$ model the symmetry is $U(N)$ and the procedure is still the same. The effective action is

$$\begin{aligned} S_{\text{eff}}[z, z^+\lambda] = \int d^2x\, dt \frac{1}{g^2} [&\partial_\mu z(x, t)\partial_\mu z^+(x, t) \\ &+ (z^+(x, t)\partial_\mu z(x, t))^2 + \lambda(x, t)(z^+(x, t)z(x, t) - 1)] \end{aligned} \tag{40}$$

and therefore

$$\begin{cases} \dfrac{\partial z_i(x, t)}{\partial t} = \dfrac{1}{g^2}[\Box z_i(x, t) - \lambda(x, t)z_i(x, t) \\ \qquad + 2(z^+(x, t)\partial_\mu z(x, t))\partial_\mu z_i(x, t)] + \eta_i(x, t) \\ \dfrac{\partial z_i^+(x, t)}{\partial t} = \dfrac{1}{g^2}[\Box z_i^+(x, t) - \lambda(x, t)z_i^+(x, t) \\ \qquad + 2(z(x, t)\partial_\mu z^+(x, t))\partial_\mu z_i^+(x, t)] + \overline{\eta}_i(x, t) \\ \dfrac{\partial \lambda(x, t)}{\partial t} = \dfrac{1}{g^2}[z^+(x, t)z(x, t) - 1] + \theta(x, t) \end{cases} \tag{41}$$

and we rapidly find

$$1 = \int \frac{d^2p}{(2\pi)^2} \frac{g^2}{p^2 + \lambda_0} \tag{42}$$

4.3. The Model of Zinn-Justin

Recently Zinn-Justin (1991) proposed a new model which has the same chiral and $U(N)$ properties as the Gross–Neveu model, and whose stochastic action is

$$S_{[\psi,\bar{\psi},\sigma]} = \int d^Dx\, dt \Big[-\bar{\psi}(x, t)(\partial\!\!\!/ + g\sigma(x, t))\psi(x, t) + \frac{1}{2}\partial_\mu\sigma(x, t)\partial_\mu\sigma(x, t) + \frac{m^2}{2}\sigma^2(x, t) + \frac{\lambda}{4!}\sigma^4(x, t)\Big] \tag{43}$$

The main interest of this model is its renormalizability in four dimensions and the fact that at the tree level, when m^2 is negative, the chiral symmetry is spontaneously broken by the σ expectation value, which gives also a mass to the fermions.

This action leads to the following Langevin equations:

$$\begin{cases} \dfrac{\partial\psi_i(x, t)}{\partial t} = \overrightarrow{\partial\!\!\!/}\, \psi_i(x, t) + g_\sigma(x, t)\psi_i(x, t) + \eta_i(x, t) \\ \dfrac{\partial\bar{\psi}_i(x, t)}{\partial t} = \bar{\psi}(x, t)\, \overleftarrow{\partial\!\!\!/} + g\sigma(x, t)\bar{\psi}_i(x, t) + \bar{\eta}_i(x, t) \\ \dfrac{\partial\sigma(x, t)}{\partial t} = (\Box - m^2)\sigma(x, t) - \dfrac{\lambda}{6}\sigma^3(x, t) + g\bar{\psi}(x, t)\psi(x, t) + \theta(x, t) \end{cases} \tag{44}$$

With the same method as above we arrive at the two equations

$$\lim_{\substack{N\to\infty \\ t\to\infty}} \left\langle \sum_{i=1}^{N} \psi_i(x, t)\bar{\psi}_i(x, t) \right\rangle_{\eta\theta} = \sum_\alpha \left\langle x \left| \left(\frac{N}{-\partial\!\!\!/ + g\sigma_0} \right)_{\alpha\alpha} \right| x \right\rangle \tag{45}$$

and

$$0 = -m^2\sigma_0 - \frac{\lambda}{6}\sigma_0^3 + g \lim_{\substack{N\to\infty \\ t\to\infty}} \left\langle \sum_{i=1}^{N} \bar{\psi}_i(x, t)\psi_i(x, t) \right\rangle_{\eta\theta} \tag{46}$$

Therefore

$$m^2\sigma_0 + \frac{\lambda}{6}\sigma_0^3 - N K_D g^2 \int \frac{d^Dp}{(2\pi)^D} \frac{\sigma_0}{p^2 + \sigma_0} = 0 \tag{47}$$

where K_D is the trace of the identity matrix.

If we rescale for convenience $g\sigma_0$ in σ_0, we recover the equation given by Zinn-Justin (1991):

$$\frac{m^2}{g}\sigma_0 + \frac{\lambda}{6g^4}\sigma_0^3 - N K_D \int \frac{d^D p}{(2\pi)^D} \frac{\sigma_0}{p^2 + \sigma_0} = 0 \tag{48}$$

5. CONCLUSION

In this paper we have used stochastic quantization for some well-known models in the large-N limit. It is shown that one can easily retrieve the gap equation of all models considered from the factorization property only. This method is less involved than Alfaro's approach and appears much simpler. It is also a more direct way to the result than the saddle point method from path integrals, because stochastic quantization enables us to extract all information about the complete quantized field theory from a classical type of evolution equation.

Clearly this method is quite pedestrian and can certainly be applied to more complex models. We have already shown elsewhere (Grandati *et al.*, 1992, 1993) that for the $O(N)\phi^4$ model with finite N, the variational stochastic solution can be built recursively by a $1/N$ expansion. We have also derived explicitly from this variational equation an integral equation from each term of the $1/N$ expansion of the mass operator. The general form of its solution is obtained analytically. With the introduction of stochastic auxiliary fields, especially by the Hubbard–Stratanovitch procedure, new and simpler investigations of variational method in stochastic quantization are feasible. In particular one may envisage the study of fermionic, constrained, and Zinn-Justin models using a variational stochastic principle (Bérard *et al.*, n.d.).

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